Review of finite approximations, Archimedean and non-Archimedean

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Model over **R**

A 1-dimensional Hamiltonian operator of the form

$$H = -\Delta + V$$

where $\Delta = \frac{d^2}{dx^2}$ is the Laplacian and V is multiplication by a function v, both regarded as operators on suitable domains in $L^2(\mathbf{R})$.
With operators P and Q defined as $(Pf)(x) = \frac{1}{i} f'(x)$ and $(Qf)(x) = xf(x)$ on suitable domains, we have $P = \mathcal{F}^{-1} Q \mathcal{F}$ where $\mathcal{F}$ is the Fourier transform:

$$\mathcal{F}f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y) e^{-ixy}\, dy\,.$$

Notice that $-\Delta = P^2 = \mathcal{F}^{-1} Q^2 \mathcal{F}$, and that

$$Q^2 f(x) = x^2 f(x) = |x|^2 f(x), \quad H = P^2 + V.$$

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Infinite discrete model over **R**

$\epsilon > 0$, $G(\epsilon) = \epsilon\mathbf{Z}$.
$(T_h f)(x) = f(x+h)$, $h \in G(\epsilon)$, f any function on $G(\epsilon)$.

$$D(\epsilon)^+ = \epsilon^{-1}(T_\epsilon - I), \quad D(\epsilon)^- = \epsilon^{-1}(I - T_{-\epsilon})$$

$$\Delta(\epsilon) = D(\epsilon)^- D(\epsilon)^+$$

On $L^2(G(\epsilon))$ the T_h are unitary, $D(\epsilon)^- = -(D(\epsilon)^+)^*$ and $\Delta(\epsilon)$ are bounded, and

$$(-\Delta(\epsilon)f, f) = ||D(\epsilon)^+ f||^2, \quad f \in L^2(G(\epsilon))$$

Hamiltonian $H(\epsilon)$ on $L^2(G(\epsilon))$: $H(\epsilon) = -\Delta(\epsilon) + V_\epsilon$, V_ϵ acting as multiplication by the restriction of V to $G(\epsilon)$.

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Finite models over **R**

$N^0 = N^0(\epsilon)$: an integer ≥ 1 depending on ϵ; $\epsilon N^0 \to \infty$ as $\epsilon \to 0$.
$N = 2N^0 + 1$. $G(\epsilon)^0 = \{r\epsilon | r = 0, \pm 1, \ldots, \pm N^0\}$.
We get various models depending on boundary conditions. We will describe the following two: periodic model and the "Schwinger model".

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Periodic model over **R**

$H(\epsilon)^{(p)} = -\Delta(\epsilon)^{(p)} + V_\epsilon$.
Identify $G(\epsilon)^0$ with the finite group $G(\epsilon)/\left(N(\epsilon)\cdot G(\epsilon)\right)$, and hence $L^2(G(\epsilon)^0)$ with the space of functions on $G(\epsilon)$ invariant under T_h, $h \in N(\epsilon)\cdot G(\epsilon)$.
$D(\epsilon)^{\pm(p)}$ are the restrictions of $D(\epsilon)^{\pm}$ to this space.

$$\Delta(\epsilon)^{(p)} = D(\epsilon)^{-(p)} D(\epsilon)^{+(p)}, \quad (-\Delta(\epsilon)^{(p)} f, f) = ||D(\epsilon)^{+(p)} f||^2.$$

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The "Schwinger model" [1] over **R**

Also a periodic model, but with different definition of Laplacian and with explicit relation between ϵ and N.

$H(\epsilon)^{(s)} = -\Delta(\epsilon)^{(s)} + V_\epsilon$.
$\epsilon = \epsilon_N = (2\pi/N)^{1/2}$.
$\mathcal{F}_\epsilon =$ the Fourier transform on $L^2(G(\epsilon)/N \cdot G(\epsilon))$
$q_\epsilon =$ multiplication by the coordinate:
$(q_\epsilon f)(x) = xf(x)$, $x \in G(\epsilon)$
$p_\epsilon = \mathcal{F}_\epsilon^{-1} q_\epsilon \mathcal{F}_\epsilon$
$\Delta(\epsilon)^{(s)} = -p_\epsilon^2$

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[1] So named because Julian Schwinger used it in his studies of finite quantum mechanics.

Remarkable accuracy of the Schwinger model

For reasons not fully understood the Schwinger model is far superior to the usual finite difference model as far as numerical results are concerned (see the Appendix for numerical results and comparison with the finite difference operator).

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In [DVV94] also a third class of finite models were considered; these were given superscript 0. The main result of that paper could then be stated as follows[2]:

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[2] More precisely, this is the main result of the functional analytic part of the paper. In the probabilistic part a somewhat stronger convergence result was optained for stochastic Hamiltonians.

Varadarajan, Varadhan, TD [DVV94]

Theorem

Let (ϵ_n) *be a sequence tending to* 0 *and* $* = p, s, 0$. *Let* $0 \leq h_1 < h_2 < \dots$ *be the eigenvalues of* H *and* T_j *the eigensubspace corresponding to* h_j. *Then:*
(i) if J *is a compact subset of* $[0, \infty)$ *not containing any eigenvalues of* H, *then no eigenvalue of* $H(\epsilon_n)^{(*)}$ *belongs to* J *if* n *is large enough*
(ii) if J *is a compact neighborhood of* h_j *not containing any* $h_i, i \neq j$, *all the eigenvalues of* $H(\epsilon_n)^{(*)}$ *that belong to* J *converge to* h_j; *if* T_{nj} *is the span of the corresponding eigenspaces,* $\dim(T_{nj}) = \dim(T_j)$ *for* n *large enough, and there is an orthonormal basis of* T_{nj} *that converges to an orthonormal basis of* T_j.

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Atomic potential: from the thesis of Erik M. Bakken

The next 13 slides, dealing with atomic potential, are based on Ch. 5 of Erik M. Bakken's PhD thesis [Bak16].

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Hamiltonian $H = -\Delta + V$ acting in $L^2(\mathbb{R}^3)$. $(Vf)(x) = v(x)f(x)$ for some suitable function v. Later v will be specialized to the radial function $v(r) = -1/r$.
$\Delta =$ Laplacian.

Finite model:
Finite grid $G(\varepsilon)$:
$\{r\varepsilon : r = -n_0, -n_0+1, ..., n_0-1, n_0\}^3$, $n = 2n_0 + 1$.
$\{e_i\}_{i=1}^3$: an orthonormal basis for $\mathbf{R}^3$.
Discrete Laplacian in the direction i:

$$\Delta_{\varepsilon,i} f(x) = \frac{f(x+\varepsilon e_i) - 2f(x) + f(x-\varepsilon e_i)}{\varepsilon^2}, \qquad x \in G(\varepsilon). \quad (1)$$

Boundary conditions: f is interpreted to be zero in the above formula at all points which are not in the grid.
Discrete Laplacian: $\Delta_\varepsilon = \sum_{i=1}^3 \Delta_{\varepsilon,i}$.
Potential on the grid: $V_\varepsilon f(x) = v_\varepsilon(x) f(x)$, v_ε: the restriction of v to the grid.

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Imbedding

Imbedding $\# : L^2(G(\varepsilon)) \to L^2(\mathbb{R}^3)$ given by

$$f^{\#} = \varepsilon^{-3/2} \sum_{x \in G(\varepsilon)} f(x)\chi_{R(x)}, \tag{2}$$

where $\chi_{R(x)}$ is the characteristic function of the set $R(x) = \{(y_1, y_2, y_3) \in \mathbb{R}^3 : x_i - \varepsilon/2 \leq y_i < x_i + \varepsilon/2,\ i = 1, 2, 3\}$. We can view operators on $L^2(G(\varepsilon))$ as operators on $L^2(\mathbb{R}^3)$ by sending an operator A to $Q_\varepsilon A Q_\varepsilon$ where Q_ε is the orthogonal projection on the image of $\#$. The projection is given by

$$Q_\varepsilon f = \sum_{x \in G(\varepsilon)} \frac{1}{\varepsilon^3} \int_{R(x)} f(x - y)\, dy \cdot \chi_{R(x)}. \tag{3}$$

When one imbeds a Hamiltonian this way, the spectrum of the Hamiltonian is preserved except that 0 becomes an eigenvalue of infinite multiplicity (whether or not 0 was an eigenvalue before the imbedding).

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Coulomb potential

We now set $v(r) = -1/r$. In this case the spectrum of the Hamiltonian $H = -\Delta + V$ is known:
Eigenvalues: $E_k = -\frac{1}{4k^2}, \quad k = 1, 2, 3, \ldots$.
E_k has multiplicity k^2.
Continuous spectrum $= [0, \infty)$.

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Definition (Coulomb-like Operator)

Let A be a self-adjoint operator which is bounded below, with discrete spectrum below a constant c, and with $\sigma_{\text{ess}}(A) = \sigma_{\text{cont}}(A) = [c, \infty)$, where $\sigma_{\text{cont}}(A)$ is the continuous spectrum of A. Then we will call A a Coulomb-like operator.

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Definition (Convergence of spectra)

Let A be a Coulomb-like operator and let A_n, $n = 1, 2, ...$ be self-adjoint operators which are bounded below, and let A_n have discrete spectrum. The eigenvalues of A are denoted by $\lambda_1 \leq \lambda_2 \leq ...$ and are counted with multiplicity. Assume that:

1. If J is a compact subset of $\mathbb{R}$ containing no eigenvalues of A, then no eigenvalues of A_n will be in J for sufficiently large n.
2. For every $\lambda \in \sigma(A)$ there exists a sequence $\lambda_n \in \sigma(A_n)$ such that $\lambda_n \to \lambda$. If $J = [a, b]$ is a compact interval with $c < a < b$, then $P_J(A_n)$ converges strongly to $P_J(A)$.
3. If J is a compact neighborhood containing the eigenvalue λ_j, and no other eigenvalues of A different from λ_j, then all the eigenvalues of A_n in J converge to λ_j. Furthermore $||P_J(A_n) - P_J(A)|| \to 0$ as $n \to \infty$.

We will then say that the spectrum of A_n converges to the spectrum of A, and we will denote it by

$$\sigma(A_n) \to \sigma(A). \tag{4}$$

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Hamiltonian in a cube

To prove convergence of the finite quantum systems we will first prove convergence inside a finite box.
The Hamiltonian in the open cube
$T_b = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : |x_i| < b, i = 1, 2, 3\}$ will be denoted by $H_b = -\Delta_b + V_b$, where Δ_b is the Laplacian, and V_b acts as multiplication by a modified version v_b of the potential function v (in order to deal with the singularity at the origin):

$$v_b(r) = \begin{cases} -\frac{1}{r} & \text{if } r > 1/b \\ -b & \text{if } r \leq 1/b. \end{cases} \tag{5}$$

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Lemma

The Hamiltonian H_b has a compact resolvent, and thus a discrete spectrum.

Lemma

The k'th eigenvalue for H_b is always bigger than or equal to the k'th eigenvalue of H, that is, $\lambda_k^b \geq \lambda_k$.

Proof.

This follows from a Max-Min theorem, by regarding $C_c^\infty(T_b)$ as a subspace of $C_c^\infty(\mathbb{R}^3)$ (by setting functions in $C_c^\infty(T_b)$ equal to zero outside T_b). □

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Since H is bounded from below, it can be made positive by adding αI to it, for some $\alpha > 0$. Also, if H is positive, so is H_b, hence the α which makes H positive, makes H_b positive, too. For this reason we will from now on assume H and H_b to be positive (although we are actually working with $H + \alpha I$ and $H_b + \alpha I$).

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The rest of the proof goes in two steps. First we show convergence of the finite models inside a finite cube, and by using bigger and bigger cubes we will in the second part show convergence of the finite models in $\mathbb{R}^3$. It will be convenient to use a different set of parameters than n and ε to describe the grid when we prove convergence inside a fixed cube. So for the open cube $T_b = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : |x_i| < b, i = 1, 2, 3\}$, let

$$\varepsilon = \frac{2b}{n+1}. \tag{6}$$

Then for the grid $G(\varepsilon)$ described above we will rather use the notation $G_{b,n}$.

We will also need to imbed $L^2(G_{b,n})$ into $L^2(T_b)$:

$$f^{\#} = \varepsilon^{-3/2} \sum_{x \in G_{b,n}} f(x)\chi_{R(x)}, \tag{7}$$

where $\chi_{R(x)}$ is the characteristic function of the set $R(x) = \{(y_1, y_2, y_3) \in T_b : x_i - \varepsilon/2 \leq y_i < x_i + \varepsilon/2, i = 1, 2, 3\}$. Note that with this imbedding, $f^{\#}(x)$ will be zero if any of the coordinates x_i is less than $\varepsilon/2$ away from the boundary. Imbedding the Hamiltonian has the same effect on the spectrum as the imbedding into $L^2(\mathbb{R}^3)$: it preserves the spectrum, except that 0 becomes an eigenvalue of infinite multiplicity. We will now change notation for all operators on the grid: Replace H_ε by $H_{b,n}$, V_ε by $V_{b,n}$, $\Delta_{\varepsilon,i}$ by $\Delta_{b,n,i}$, Δ_ε by $\Delta_{b,n}$, and Q_ε by $Q_{b,n}$. We will also use $v_{b,n}$ instead of v_ε, but $v_{b,n}$ is now the restriction of the above defined v_b (instead of v) to the grid. As we did for H and H_b, we will write $H_{b,n}$ for $H_{b,n} + \alpha I$.

Lemma

$H_{b,n}$ converges strongly to H_b on $C_0^\infty(\bar{T}_b)$ as $n \to \infty$.

Lemma

Let $K_{b,n} = (I + H_{b,n})^{-1}$, and let $g_{b,n} \in L^2(G_{b,n})$ with $||g_{b,n}||_{L^2(G_{b,n})} \leq 1$. Also let $f_{b,n} = K_{b,n} g_{b,n}$. Then the sequence $(f_{b,n}^{\#})_n$ is relatively compact in $L^2(T_b)$.

Using the two previous lemmas, the next theorem follows from Lemma 2.3 and Theorem 2.4 in [DVV94].

Theorem

With H_b and $H_{b,n}$ defined as above, $\sigma(H_{b,n}) \to \sigma(H_b)$ as $n \to \infty$, where the spectral convergence is in the sense of the definition on slide 18.

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Now we are ready to start proving that there is a sequence of finite Hamiltonians $H_{b,n(b)}$ such that $\sigma(H_{b,n(b)}) \to \sigma(H)$ as $b \to \infty$.

Lemma

Let $n(b)/b \to \infty$ as $b \to \infty$. Then the operators $H_{b,n(b)}$ converge strongly to H on the common core $C_c^\infty(\mathbb{R}^3)$ as $b \to \infty$.

Lemma

For any sequence $\delta_b > 0$ such that $\delta_b \to 0$ as $b \to \infty$, there exists a sequence $n(b)$ such that $n(b)/b \to \infty$ and $\lambda_k^{b,n(b)} \geq \lambda_k - \delta_b$ for $k \leq b$.

Lemma

Let A_n be a sequence of self-adjoint operators with discrete spectrum converging strongly to a Coulomb-like operator A on a common core. If there exists a sequence $\delta_n \to 0$ such that $\lambda_k^n \geq \lambda_k - \delta_n$ for $k \leq n$, then $\sigma(A_n) \to \sigma(A)$.

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Now we can use this lemma to get the main theorem. Notice that the imbedded versions of the finite-dimensional operators do not have purely discrete spectrum because 0 is in the essential spectrum, but this does not matter since it comes after the discrete spectrum of H and is a part of the essential spectrum of H.

Theorem (Theorem 4.3 in [Bak16])

Let as before H be the Coulomb Hamiltonian, and $H_{b,n(b)}$ the finite-dimensional Hamiltonians associated to H. Let δ_b be a sequence converging to 0 and let $n(b)/b$ go to infinity as $b \to \infty$ such that $\lambda_k^{b,n(b)} \geq \lambda_k - \delta_b$ for $k \leq b$. Then $\sigma(H_{b,n(b)}) \to \sigma(H)$ as $b \to \infty$.

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Local fields I

A local field is a non-discrete, locally compact field. The only connected local fields are $\mathbf{R}$ and $\mathbf{C}$. Disconnected local fields are, in fact, totally disconnected.

Every local field comes equipped with a canonical absolute value which defines its topology. It is is induced by the Haar measure, and is Archimedean in the case of $\mathbf{R}$ and $\mathbf{C}$, and non-Archimedean in all other cases. It coincides with the usual absolute values for the fields $\mathbf{R}$, $\mathbf{C}$, and $\mathbf{Q}_p$.

For a general local field K we will denote the canonical absolute value by $|\cdot|$ (for $\mathbf{Q}_p$ we will still denote it by $|\cdot|_p$).

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Local fields II

Convention

From now on the term 'local field' means a (totally) disconnected, non-discrete, locally compact field.

Let K be a local field with canonical absolute value $|\cdot|$. Define

$$O = \{x \in K : |x| \leq 1\}, \quad P = \{x \in K : |x| < 1\}, \quad U = O \setminus P.$$

O: a compact subring of K (the *ring of integers*). It is a discrete valuation ring, i.e., a principal ideal domain with a unique maximal ideal.
P: the unique non-zero maximal ideal of O, called the *prime ideal*. Any element $\beta \in P$ such that $P = \beta O$ is called a *uniformizer* (or a *prime element*) of K. For $\mathbf{Q}_p$ one can choose $\beta = p$.

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Local fields III

$U = O \setminus P$: *The group of units* of O.
O/P: a finite field with $q = p^f$ elements (p: a prime number, f: a natural number).

If β is a uniformizer, then $|\beta| = 1/q$. Range$(|\cdot|) = \{q^N : N \in \mathbf{Z}\}$.

If S is a complete set of representatives for the residue classes in O/P, every non-zero element $x \in K$ can be written uniquely in the form:

$$x = \beta^{-m}(x_0 + x_1\beta + x_2\beta^2 + \cdots),$$

where $m \in \mathbf{Z}$, $x_j \in S$, $x_0 \notin P$. With x written in this form, we have $|x| = q^m$.

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Local fields IV

Theorem (Classification of local fields)

Characteristic zero. *Every local field of characterisitic zero is a finite extension of* $\mathbf{Q}_p$ *for some* p.

Positive characteristic. *Every local field of positive characteristic* p *is isomorphic to the field* $\mathbf{F}_q((t))$ *of Laurent series over a finite field* $\mathbf{F}_q$, *where* $q = p^f$ *for some integer* $f \geq 1$.

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Model over K

We use the complex Hilbert space $L^2(K)$ with respect to Haar measure on K.
The operators P and Q cannot be defined as in the real case since in general we cannot multiply a complex number by an element of K.
However, in the real case the Laplacian can also be expressed as $\Delta = -\mathcal{F}^{-1}Q^2\mathcal{F}$. In the local field case it is therefore natural to define the Laplacian as:

$$\Delta = -\mathcal{F}^{-1}Q^2\mathcal{F}$$

where now

$$Q^2 f(x) = |x|_p^2 f(x).$$

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With $P = \mathcal{F}^{-1} Q \mathcal{F}$ we have $\Delta = -P^2$, and for the Hamiltonian we take

$$H = -\Delta + V = P^2 + V\,,$$

where the potential V is given by $(Vf)(x) = v(x)f(x)$ for a suitable function v on K ($f \in L^2(K)$, $x \in K$).
In the local field setting it is customary to study $H = P^\alpha + V$ for any $\alpha > 0$, since the qualitative behavior of P^α and H does not change with $\alpha > 0$. So we finally take

$$H = P^\alpha + V$$

as our object of study.

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Finite model over K

Let n be a natural number, set $B_n = H_{-n} = \beta^{-n}O$ (ball of radius q^n) and $G_n = H_{-n}/H_n$.
G_n is a finite cyclic group with q^{2n} elements and with generator $= q^{-n} + H_n$.
Since H_n is an open subset of K, we obtain a Haar measure μ_n on $G_n = H_{-n}/H_n$ from the Haar measure μ of K by setting $\mu_n(x + H_n) = \mu(x + H_n) = \mu(H_n) = q^{-n}$, for $x + H_n \in G_n = H_{-n}/H_n$.
So each point of G_n has mass $= q^{-n}$, and the total mass of G_n is $q^{2n} \cdot q^{-n} = q^n$.
Each element of G_n has a unique representative of the form $a_{-n}\beta^{-n} + a_{-n+1}\beta^{-n+1} + \cdots + a_{-1}\beta^{-1} + a_0 + a_1\beta + \cdots + a_{n-2}\beta^{n-2} + a_{n-1}\beta^{n-1}$, $a_i \in S$. We denote this set by X_n, and call it *the canonical set of representatives* for G_n; we also give it the group structure coming from its natural identification with G_n.

Imbedding

With the above choice of Haar measure on G_n the mapping which sends the characteristic function of the point $x + H_n$ in G_n to the characteristic function of the subset $x + H_n$ in K, becomes an isometric imbedding of $L^2(G_n)$ into $L^2(K)$.

Important subspaces of $L^2(K)$:
$\mathcal{C}_n = \{f \in L^2(K) | \operatorname{supp}(f) \subset B_n\}$
$\mathcal{S}_n = \{f \in L^2(K) | f \text{ is locally constant of order} \leq p^{-n}\}$
$\mathcal{D}_n = \mathcal{C}_n \cap \mathcal{S}_n$
We have: $\mathcal{F}\mathcal{C}_n = \mathcal{S}_n$ and $\mathcal{F}\mathcal{S}_n = \mathcal{C}_n$, so $\mathcal{F}\mathcal{D}_n = \mathcal{D}_n$.

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Commutation rules

Let C_n and S_n be the orthogonal projections on $\mathcal{C}_n$ and $\mathcal{S}_n$, respectively:
$C_n f = f \cdot \mathbf{1}_{B_n}$, $S_n f(x) = q^n \int_{H_n} f(x+y) d_n(y)$.

C_n and S_n commute, so $D_n = C_n S_n$ is the orthogonal projection on $\mathcal{D}_n$.
Furthermore, D_n commutes with the Fourier transform $\mathcal{F}$ on K, and $\mathcal{F} D_n = D_n \mathcal{F}$ coincides with the finite Fourier transform $\mathcal{F}_n$ on G_n.

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Dynamical operators for the finite model

Functions on the finite grid G_n can be identified with functions on K which have support in B_n and are invariant under translation by elements of H_n. For the position operator Q_n on $L^2(G_n)$ we take the restriction to these functions of the position operator Q on $L^2(K)$: $(Qf)(x) = |x|_p f(x)$. The momentum operator $P_n = \mathcal{F}_n^{-1} Q_n \mathcal{F}_n$ then becomes the restriction of $P = \mathcal{F}^{-1} Q \mathcal{F}$ to $L^2(G_n)$. For our finite Hamiltonian we take[3]

$$H_n = P_n^\alpha + V_n$$

[3] See slide 33 for a comment on the exponent α.

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Uniform compactness

Definition (Uniform compactness)

A sequence of bounded operators (M_n) on a Hilbert space $\mathcal{H}$ is said to satisfy a condition of *uniform compactness* [4] if the following conditions hold:

1. The sequence (M_n) is uniformly bounded.
2. There are subspaces L_n with L_n invariant under M_n such that for every sequence (g_n) with $g_n \in L_n$ and $||g_n|| \leq 1$, the sequence $(M_n g_n)$ is relatively compact in $\mathcal{H}$.

[4] Notice that the individual operators M_n are not required to be compact on the whole space $\mathcal{H}$ (and in our applications they will not be). Still, if the above conditions are fulfilled, we will say that the sequence (M_n) is uniformly compact, even if each M_n is compact only on the subspace M_n.

For our purposes the usefulness of uniform compactness lies in the following two results. They give a strong connection between the spectral data of the approximating operators (M_n) and their strong limit M.

Lemma

Let M_n, L_n be as in the definition above, and assume that the sequence M_n converges strongly to a bounded operator M. Assume further that there are eigenvectors g_n and corresponding eigenvalues λ_n such that $g_n \in L_n$, $||g_n|| = 1$ and $M_n g_n = \lambda_n g_n$. Then any non-zero cluster point λ_0 of the sequence (λ_n) is an eigenvalue of M, and there is a subsequence of (g_n) which converges to a vector g such that $Mg = \lambda_0 g$.

Proposition (Cfr. Lemma 3 in [DVV94])

Keep the notation and assumptions of the previous lemma. In addition, assume the following: (i) The operators M_n, M are self-adjoint, and $0 \leq M, M_n \leq I$, (ii) M is compact on $\mathcal{H}$, and M_n is compact on L_n. Then the following hold:

1. *If J is a compact subset of $(0, 1]$ with $J \cap \sigma_p(M) = \emptyset$, then $J \cap \sigma_p(M_n) = \emptyset$ for large n.*
2. *If $\lambda \in \sigma_p(M)$, there exists a sequence (λ_n) with $\lambda_n \in \sigma(M_n)$ such that $\lambda_n \to \lambda$. Further, if J is a compact neighborhood of an eigenvalue $\lambda \in \sigma_p(M)$, not containing any other eigenvalues of M, then any sequence (λ_n) with $\lambda_n \in \sigma_p(M_n) \cap J$ converges to λ.*
3. *Let λ and J be as in (2). Then $\dim P^{M_n}(J) = \dim P^M(J)$ for large n, and for each orthonormal basis $\{e_1, \ldots, e_m\}$ for $r\left(P^M(J)\right)$ there is, for each n, an orthonormal basis $\{e_1^n, \ldots, e_m^n\}$ for $r\left(P^{M_n}(J)\right)$ such that $\lim_{n\to\infty} e_i^n = e_i$, $i = 1, \ldots, m$.*

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Proposition

With $M_n = (I + H_n)^{-1}$, $L_n = \mathcal{D}_n \simeq L^2(G_n)$, and $\mathcal{H} = L^2(K)$, the resolvents $(I + H_n)^{-1}$ are uniformly compact in the sense of Definition 14.

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The analog of the main convergence theorem in [DVV94] can now be established for a general local field in [BD15]:

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Main convergence theorem[5]

Analog of Theorem 4 in [DVV94]

Theorem

1. *If J is a compact subset of $[0,\infty)$ with $J \cap \sigma(H) = \emptyset$, then $J \cap \sigma(H_n) = \emptyset$ for large n.*
2. *If $\lambda \in \sigma(H)$, there exists a sequence (λ_n) with $\lambda_n \in \sigma(H_n)$ such that $\lambda_n \to \lambda$. Further, if J is a compact neighborhood of an eigenvalue $\lambda \in \sigma(H)$, not containing any other eigenvalues of H, then any sequence λ_n with $\lambda_n \in \sigma(H_n) \cap J$ converges to λ.*
3. *Let λ and J be as in (2). Then $\dim P^{H_n}(J) = \dim P^H(J)$ for large n, and for each orthonormal basis $\{e_1, \ldots, e_m\}$ for $\mathrm{Ran}\left(P^H(J)\right)$ there is, for each n, an orthonormal basis $\{e_1^n, \ldots, e_m^n\}$ for $\mathrm{Ran}\left(P^{H_n}(J)\right)$ such that $\lim_{n\to\infty} e_i^n = e_i$, $i = 1, \ldots, m$.*

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[5] $\sigma(A)$ = spectrum of A. P^A = spectral measure of A. $\mathrm{Ran}(A)$ = range of A.

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In [DVV94] two proofs of the main theorem were given: a functional analytic one and a stochastic one. The stochastic proof gave a stronger convergence result for the eigenfunctions (uniform convergence on compacta). The proof in [BD15] used functional analytic methods. It is of interest to give a stochastic proof also in the non-Archimedean case, and this has been accomplished in [BDW17]. We now give a review of the results in [BDW17].

Brownian motion and the heat equation over **R** I

Brownian motion is described by a family of Wiener measures $(W_x)_{x\in\mathbf{R}}$, which in turn are generated by the probability densities $p_t(x) = \frac{1}{\sqrt{2t}} e^{-x^2/4t}$ in the following sense:

$$\int_{C([0,\infty):\mathbf{R})} f(\omega(t))dW_x(\omega) = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} f(y)p_t(x-y)dy$$

for all "observables" f belonging to a suitable class of functions on **R**. The function $u(x,t) = p_t(x)$ is a fundamental solution of the heat equation

$$\frac{\partial u}{\partial t}(x,t) = \Delta u(x,t) \tag{8}$$

Brownian motion and the heat equation over **R** II

which by Fourier transform becomes

$$\frac{\partial \hat{u}}{\partial t}(\xi, t) = -\xi^2 \hat{u}(\xi, t) \tag{9}$$

and so

$$\hat{p}_t(\xi) = \hat{u}(\xi, t) = e^{-t\xi^2}, \tag{10}$$

taking into account that $p_t(x)$ is a fundamental solution. The $(p_t)_{t>0}$ form a semi-group under convolution, and thus give rise to a semi-group of operators $(T_t)_{t>0}$ by $T_t f = p_t * f$. The infinitesimal generator of $(T_t)_{t>0}$ is the Laplacian Δ (on a suitable domain), so we can also write $e^{t\Delta} f = p_t * f$.

Brownian motion and the heat equation over K I

Over a local field K we still let t be a positive real parameter, but the role of the Laplacian Δ is played by the operator $-P^\alpha$ (remember that $\Delta = -P^2$ over $\mathbf{R}$), and so the heat equation (8) becomes

$$\frac{\partial u}{\partial t}(x,t) = -(P^\alpha u)(x,t), \quad \text{i.e.: } \frac{\partial u}{\partial t}(x,t) = -(\mathcal{F}^{-1}|\cdot|^\alpha \mathcal{F} u)(x,t) \tag{11}$$

thus

$$\frac{\partial \hat{u}}{\partial t}(\xi,t) = -|\xi|^\alpha \hat{u}(\xi,t) \tag{12}$$

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Brownian motion and the heat equation over K II

giving

$$\hat{u}(\xi, t) = e^{-t|\xi|^{\alpha}} \tag{13}$$

by a similar normalization as above. In analogy with the real case one now defines

$$p_t(x) = (\mathcal{F}^{-1}(e^{-t|\cdot|^{\alpha}}))(x) = \int_K e^{-t|\xi|^{\alpha}} \chi(x\xi)\, d\xi. \tag{14}$$

The $(p_t)_{t>0}$ again form a semi-group under convolution (since clearly $(\hat{p}_t)_{t>0}$ form a semi-group under multiplication), and $\int_K p_t(x)dx = 1$ for all $t > 0$ (since $\hat{p}_t(0) = 1$ for all $t > 0$). Thus the only thing missing for the $(p_t)_{t>0}$ to generate a Wiener measure as above, is the positivity of the $(p_t)_{t>0}$. And this has been proved by several authors in various settings (see [Koc01, Ch. 4] and references therein, and [Var97]).

Brownian motion and the heat equation at the finite level I

For our finite model we pursue the above analogy and define

$$p_{t,n}(x) = (\mathcal{F}_n^{-1} e^{-t|\cdot|^\alpha})(x), \tag{15}$$

in analogy with (14). Here we regard $e^{-t|\cdot|^\alpha}$ as a function on X_n. We still have

$$e^{-tP_n^\alpha} f = p_{t,n} * f \tag{16}$$

since

$$\begin{aligned}(e^{-tP_n^\alpha} f)(x) &= (e^{-t\mathcal{F}_n^{-1} Q_n^\alpha \mathcal{F}_n} f)(x) = (\mathcal{F}_n^{-1} e^{-tQ_n^\alpha} \mathcal{F}_n f)(x) \\ &= (\mathcal{F}_n^{-1}(e^{-t|\cdot|^\alpha} \mathcal{F}_n f))(x) = (\mathcal{F}_n^{-1}(e^{-t|\cdot|^\alpha}) * f)(x) \\ &= (p_{t,n} * f)(x),\end{aligned}$$

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Brownian motion and the heat equation at the finite level II

where the convolution $*$ now is over X_n:

$$(f * g)(x) = \int_{X_n} f(y)g(x-y)d\mu_n(y) = q^{-n} \sum_{y \in X_n} f(y)g(x-y).$$

The one-parameter family $(p_{t,n})_{t>0}$ is a semi-group under convolution (since clearly $(\hat{p}_{t,n})_{t>0}$ is a multiplicative semi-group), and $\int_{X_n} p_{t,n}(x)dx = 1$ for all n and for all $t > 0$ (since $\hat{p}_{t,n}(0) = 1$). It remains to show that the $p_{t,n}$ are positive:

Lemma

We have $p_{t,n}(x) > 0$ for all $x \in X_n$, all n and all $t > 0$, hence $(p_{t,n})_{t>0}$ defines a probability distribution over X_n.

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Probability measures on the space of Skorokhod functions

From now on we will work on a fixed time interval $[0, t]$.
The space $D[0, t]$ of Skorokhod functions are the functions defined on $[0, t]$ with values in K which satisfy the following two criteria:

1. For each $s \in (0, t)$, $f(s \pm 0)$ exist; $f(0+0)$ and $f(t-0)$ exist.
2. $f(s+0) = f(s)$ for all $s \in [0, t)$, and $f(t) = f(t-0)$.

We will use the densities $p_{s,n}$ to construct, for each n and for each $a \in X_n$, a probability measure $\mathbf{P}_a^n$ on the space $D[0, t]$, and subsequently show that these measures converge weakly to the measure $\mathbf{P}_a$ on $D[0, t]$ which is constructed from p_s. The measure $\mathbf{P}_a^n$ will give full measure to the paths which take values in the grid X_n.

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Construction of measures on the space of Skorokhod functions I

Pick a point $a \in X_n$, fix N time points $t_1 < t_2 < \cdots < t_N$, and for each $i = 1, \ldots N$, pick a Borel subset J_i of K. We define a measure $\mathbf{P}_a^n$ on the cylinder sets $\{\omega : [0, t] \to K : \omega(t_i) \in J_i\}$ by

$$\mathbf{P}_a^n(\omega(t_i) \in J_i) \tag{17}$$

$$= \sum_{b_i \in J_i \cap X_n,\, 1 \le i \le N} p_{t_1,n}(b_1 - a) \cdots p_{t_N - t_{N-1},n}(b_N - b_{N-1}) q^{-nN}. \tag{18}$$

By Kolmogorov's Extension Theorem [Øks98, Thm. 2.1.5], $\mathbf{P}_a^n$ has a unique extension to a probability measure on $\Omega[0, t]$, the space of all functions $\omega : [0, t] \to K$, equipped with the σ-algebra generated by all cylinder sets. To get a probability measure on $D[0, t]$, equipped with the Borel sets coming from the Skorokhod topology, we need to check the Čentsov

Construction of measures on the space of Skorokhod functions II

criterion, which says: If there are constants $\alpha, \beta, \gamma, C > 0$ such that

$$E_{\mathbf{P}_a^n}(|Y_{t_1} - Y_{t_2}|^\alpha |Y_{t_2} - Y_{t_3}|^\beta) \leq C|t_1 - t_3|^{1+\gamma} \tag{19}$$

for all $0 < t_1 < t_2 < t_3$, then there is a unique measure on $D[0, t]$ which satisfies the condition (17). Here $E_{\mathbf{P}_a^n}$ denotes the expectation w.r.t. the measure $\mathbf{P}_a^n$, and Y_s denotes the random variable $Y_s(\omega) = \omega(s)$, $\omega \in \Omega[0, t]$, $s \in [0, t]$. The random variables Y_s define a process with independent increments with respect to each of the measures $\mathbf{P}_a^n$.

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Weak Convergence of Unconditioned Measures

Theorem

Let $a_n \in X_n$, $a \in K$ be such that $a_n \to a$ as $n \to \infty$. Then

$$\mathbf{P}^n_{a_n} \Rightarrow \mathbf{P}_a \text{ as } n \to \infty$$

where $\Rightarrow$ denotes weak convergence of measures.

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Conditioned measures

Let $a, b \in X_n$. We define the conditioned measure $\mathbf{P}^n_{a,b,t}$ of a Borel set $A \subset D[0,t]$ by[6]

$$\mathbf{P}^n_{a,b,t}(A) = \frac{\mathbf{P}^n_a(A \cap (\omega(t) = b))}{\mathbf{P}^n_a(\omega(t) = b)}. \tag{20}$$

Let a_n and b_n be sequences of grid points converging to a and b respectively. Then for time points $t_1, ..., t_N$ in $[0,t]$ and balls J_i in K

$$\mathbf{P}^n_{a_n,b_n,t}(\omega(t_i) \in J_i) = \frac{\mathbf{P}^n_{a_n}((\omega(t_i) \in J_i) \cap (\omega(t) = b_n))}{\mathbf{P}^n_{a_n}(\omega(t) = b_n)}. \tag{21}$$

[6] Here and in the following we use the probabilist's notation for sets: $(\omega(t) = b)$ is a shortcut notation for the set $\{\omega : \omega(t) = b\}$.

Weak Convergence of Conditioned Measures

Theorem

If $a_n \in X_n \to a \in K$ and $b_n \in X_n \to b \in K$, then $\mathbf{P}^n_{a_n,b_n,t} \Rightarrow \mathbf{P}_{a,b,t}$. The convergence is uniform when (a, b) varies in compact subsets of $K \times K$.

Theorem

For each $a \in X_n$ the measure $\mathbf{P}^n_a$ gives full measure to the paths on the grid, that is,

$$\mathbf{P}^n_a(\omega : \omega(s) \in X_n, \forall s \in [0, t]) = 1 \tag{22}$$

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Feynman-Kac formulas I

For the Hamiltonian H over K:

$$(e^{-tH} f)(x) = \int_K K_t(x,y) f(y)\, dy, \quad f \in L^2(K)\,, \tag{23}$$

where

$$K_t(x,y) = \int_{D[0,t]} e^{-\int_0^t v(\omega(s))\, ds}\, d\mathbf{P}_{x,y,t}(\omega) \cdot p_t(y-x)\,. \tag{24}$$

Feynman-Kac formulas II

At the finite level we can similarly prove:

Theorem (Feynman-Kac at the finite level)

$$\begin{aligned}(e^{-tH_n}f)(x) &= \int_{X_n} K_t^n(x,y)f(y)\,d\mu_n(y) \\ &= q^{-n}\sum_{y\in X_n} K_t^n(x,y)f(y), \quad f\in L^2(X_n)\end{aligned} \tag{25}$$

where

$$K_t^n(x,y) = \int_{D[0,t]} e^{-\int_0^t v_n(\omega(s))\,ds}\, d\mathbf{P}^n_{x,y,t}(\omega)\cdot p_{t,n}(y-x), \quad x,y\in X_n\,. \tag{26}$$

Convergence of propagators

Lemma

K_t^n converges continuously to K_t, i.e., if $x_n \in X_n \to x \in K$ and $y_n \in X_n \to y \in K$ as $n \to \infty$, then

$$K_t^n(x_n, y_n) \to K_t(x, y).$$

In particular, K_t^n converges uniformly to K_t on compact sets.

The proof uses the Feynman-Kac formula (21).

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Stochastic proof of main theorem

Theorem

For any $t > 0$,

$$\mathrm{Tr}(e^{-tH_n}) \to \mathrm{Tr}(e^{-tH}) \tag{27}$$

$$||e^{-tH_n} - e^{-tH}||_1 \to 0 \tag{28}$$

as $n \to \infty$.

Convergence in trace norm implies convergence in operator norm which gives convergence of eigenvalues and eigenfunctions (see pp. 289-290 in [RS80]). Thus we have reproved by stochastic methods the main convergence theorem (16).

However, the stochastic method gives an even stronger convergence result, namely uniform convergence on compacta of the eigenfunctions:

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Uniform Convergence on Compacta of Eigenfunctions

Theorem

Let $f_{n,j}$ and f_j be eigenfunctions of H_n and H corresponding to the eigenvalues $\lambda_{n,j}$ and λ_j respectively. Assume that $\lambda_{n,j}$ converges to λ_j and that $f_{n,j}$ converges to f_j in $L^2(K)$. Then

$$f_{n,j} \to f_j \tag{29}$$

uniformly on compacta.

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Outline

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The Schwinger model

In order to illustrate the remarkable accuracy of the Schwinger model, we present on the following pages some numerical results for the harmonic oscillator with $N = 5, 21$, and 81 points in the grid.The finite eigenfunctions and the Hermite functions are rendered in the same diagram, and, as can be seen, the finite eigenfunctions lie smack on the Hermite functions, already at $N = 5$. Also, the eigenvalues show good agreement already at $N = 5$, and at $N = 81$ the first 11 eigenvalues are exact up to machine accuracy.

We also present some tables where we compare the Schwinger model with the finite difference model, showing clear superiority of the former over the latter.

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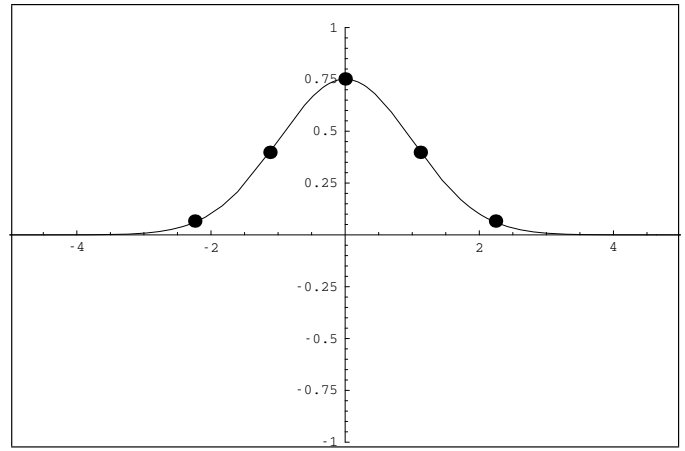
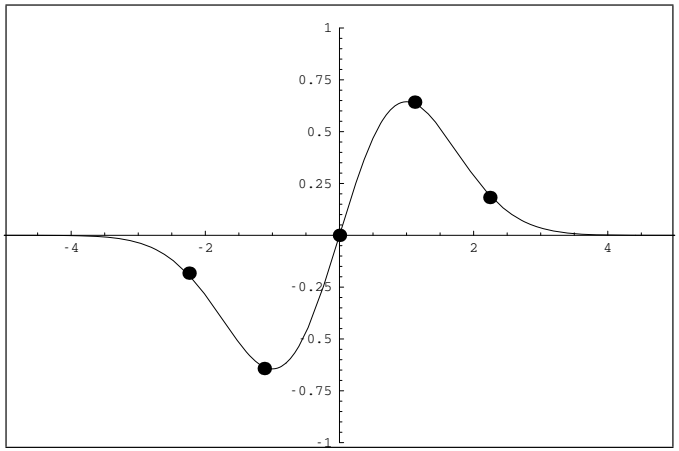
Schwinger eigens with $N = 5$

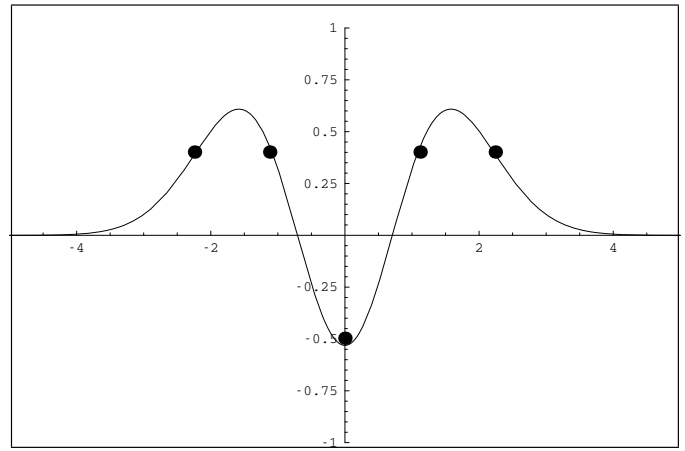
Eigenvalues for H_5
0.4969786369997017
1.538153655416401
2.273277799898967
3.512928870280916
4.745031651763187



Eigenfunction no. 0
for H_5 (dotted) and H (smooth).



Eigenfunction no. 1
for H_5 (dotted) and H (smooth).



Eigenfunction no. 2
for H_5 (dotted) and H (smooth).

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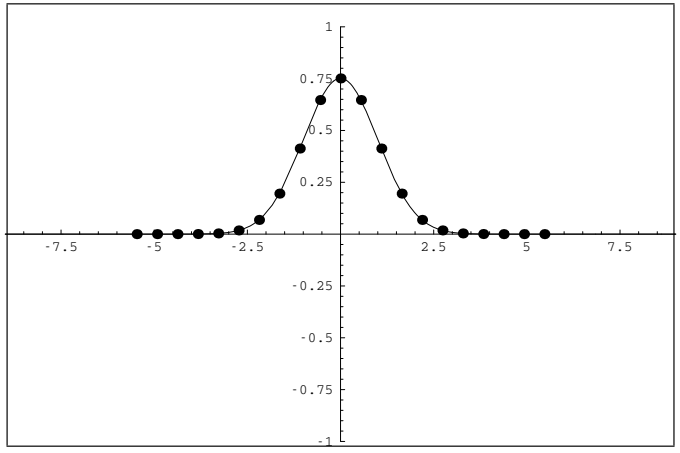
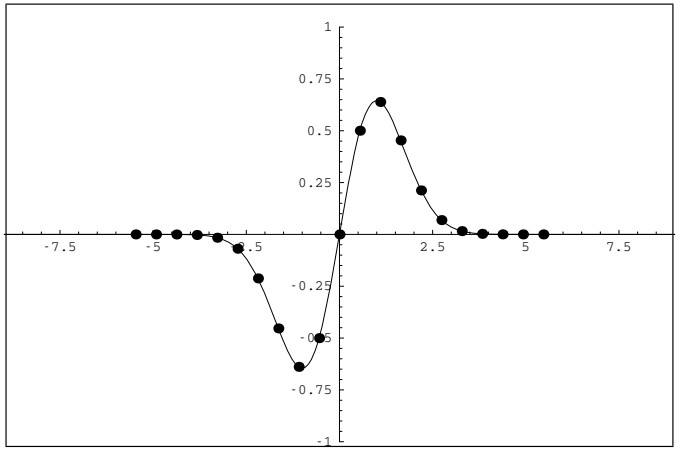
Schwinger eigens with $N = 21$

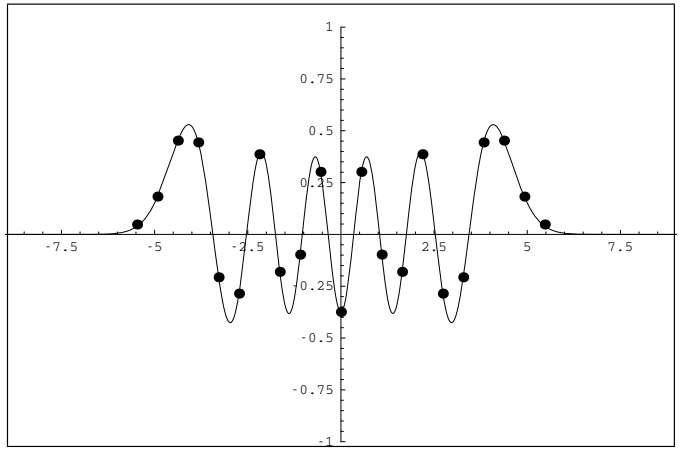
Eigenvalues for H_{21}
0.4999999999999396
1.50000000000396
2.499999999873056
3.500000002436515
4.499999963136251
5.500000389175935
6.499996311530578
7.500024950572093
8.499832644686019
9.500769902078664
10.49608851334482



Eigenfunction no. 0
for H_{21} (dotted) and H (smooth).



Eigenfunction no. 1
for H_{21} (dotted) and H (smooth).



Eigenfunction no. 10
for H_{21} (dotted) and H (smooth).

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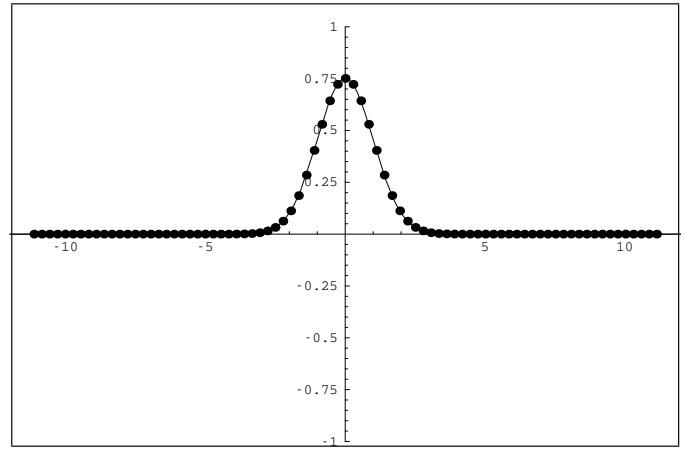
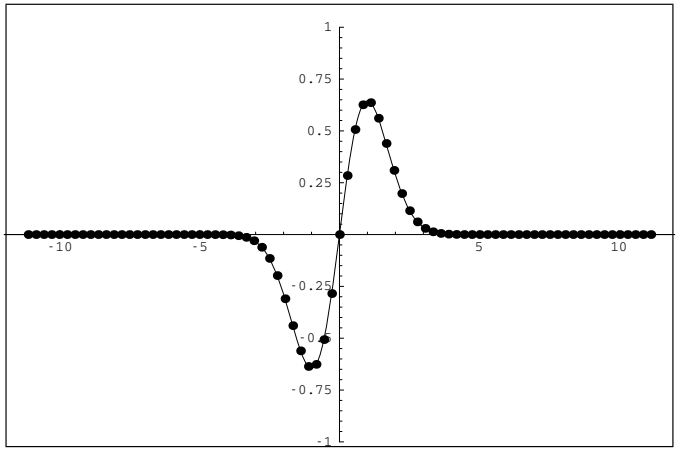
Schwinger eigens with $N = 81$

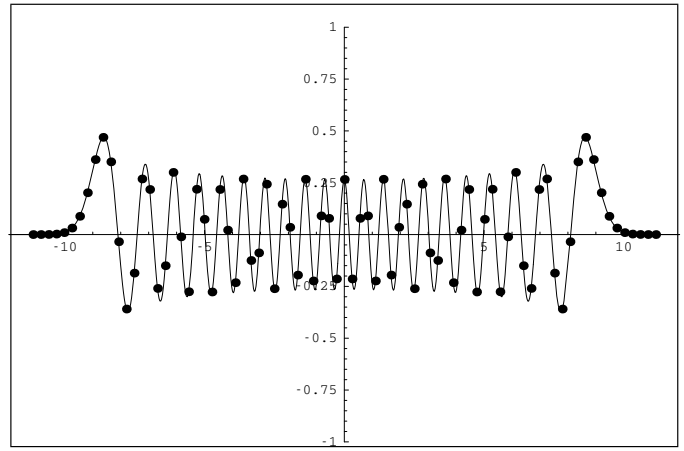
Eigenvalues for H_{81}
0.500000000000002
1.500000000000014
2.500000000000016
3.499999999999992
4.499999999999959
5.5
6.500000000000021
7.500000000000014
8.499999999999976
9.500000000000012
10.5



Eigenfunction no. 0
for H_{81} (dotted) and H (smooth).



Eigenfunction no. 1
for H_{81} (dotted) and H (smooth).



Eigenfunction no. 40
for H_{81} (dotted) and H (smooth).

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Comparison of Schwinger model and finite difference model, $N = 81$

N=81, 7 decimals

Exact	Schwinger	Finite diff.
1/2	0.5000000	0.4975640
3/2	1.5000000	1.4877712
5/2	2.5000000	2.4680608
7/2	3.5000000	3.4382768
9/2	4.5000000	4.3982546
11/2	5.5000000	5.3478205
13/2	6.5000000	6.2867905
15/2	7.5000000	7.2149698
17/2	8.5000000	8.1321509
19/2	9.5000000	9.0381131
21/2	10.5000000	9.9326202

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Comparison of Schwinger model and finite difference model, $N = 241$

N=241, 7 decimals

Exact	Schwinger	Finite diff.
1/2	0.5000000	0.4991839
3/2	1.5000000	1.4959143
5/2	2.5000000	2.4893615
7/2	3.5000000	3.4795090
9/2	4.5000000	4.4663402
11/2	5.5000000	5.4498380
13/2	6.5000000	6.4299851
15/2	7.5000000	7.4067638
17/2	8.5000000	8.3801562
19/2	9.5000000	9.3501440
21/2	10.5000000	10.3167088

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Coulomb potential

Table: Numerical result with $n = 350$ and $\varepsilon = (2\pi/n)^{1/2}$

Exact values	Numerical values
-1	-0.9814558
-1/4	-0.2505890, -0.2505890,-0.2505890, -0.2483674
-1/9	-0.11136023,-0.11136023, -0.11136023, -0.11121653, -0.11121653, -0.1111883, -0.1111883,-0.1111883, -0.1106982
-1/16	-0.06260630, -0.06260630, -0.06260630, -0.06254948, -0.06254948,-0.06254293, -0.06254293, -0.06254293, -0.06252377, -0.06252377, -0.06252377, -0.06252227, -0.06252227, -0.06252227, -0.06251815, -0.06231906

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Numerical results for the Schrödinger operator over $\mathbf{Q}_3[\sqrt{3}]$

Let as before $H = P^\alpha + V$ denote the Schrödinger operator over a local field K. The eigenfunctions of H can be divided into two main types, corresponding to two complementary subspaces of $L^2(K)$: those which are supported on a single spherical shell (which we shall call shell functions), and those which are radial[7]. Of these, only the shell functions are completely understood: They belong to eigenvalues which can be determined from Diophantine equations, and there are explicit formulae for them. For radial eigenfunctions no such explicit formulae seem to be known.

[7] With terminology as in [VVZ94], the set of shell functions comprises all the type I functions plus the shell functions of type II; the radial functions are all of type II.

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In this numerical study we specialize to the case of the Schrödinger operator $H = \frac{1}{2}(P^2 + Q^2)$ of the harmonic oscillator over the local field $\mathbf{Q}_3[\sqrt{3}]$, which is a quadratic and totally ramified extension of $\mathbf{Q}_3$. We were interested in the following questions:

- Do eigenfunctions of both types (shell functions and radial functions) show up already at the finite level?
- Is there good agreement between the theoretical and numerical eigenvalues?
- Is there good agreement between the theoretical and numerical eigenfunctions?
- Are multiplicities correct?

The answer to all these questions was 'yes'. To illustrate this, we sum up some of the results in Table 2.

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The extension $\mathbf{Q}_3[\sqrt{3}]/\mathbf{Q}_3$ is totally ramified, so with standard notation we have $e = 2$, and hence $f = 1$ since $ef = [\mathbf{Q}_3[\sqrt{3}] : \mathbf{Q}_3] = 2$. Further, from $q = p^f$ follows $q = p = 3$, and as uniformizer we can take $\beta = \sqrt{3}$, hence $|\beta| = 1/q = 1/3$. For the exponent of the different we have $d = 1$, so the canonical character χ associated with these data becomes

$$\chi(x) = \exp\left(2\pi i\{\mathrm{Tr}_{\mathbf{Q}_3[\sqrt{3}]/\mathbf{Q}_3}(\sqrt{3}^{-1}x)\}\right),\ x \in \mathbf{Q}_3[\sqrt{3}].$$

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For the finite model we did experiments with $n = 1, 2, 3, 4$, so we were working with finite grids of sizes $|X_1| = 9$, $|X_2| = 9^2 = 81$, $|X_3| = 9^3 = 729$, and $|X_4| = 9^4 = 6561$, respectively. Of particular interest to us was how the eigenfunctions came out: Would they clearly exhibit characteristics as shell functions or radial functions? They did. To illustrate this we give in the last table an excerpt from the value tables of three eigenfunctions: one is radial, one is a linear combination of two shell functions, and one is a pure shell function. We also wanted to compare our numerically computed eigenfunctions to the theoretical ones (evaluated on the grid). To do this, we measured the distance from each of the former to the linear span of the latter. Up to machine accuracy (10^{-16}), the distance came out as zero. We find this quite remarkable.

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The following two tables should be self-explanatory[8]. The data are taken from a computer run with $n = 2$ (i.e., 81 points in the finite grid). Each of the functions in the last table is represented with 28 values, with values coming from each of the 5 shells which occur for $n = 2$.

[8] In the estimate for the lowest eigenvalue in Table 2 (first entry in column 1) we are assuming that the estimate given in [VVZ94, p. 190] is valid also in our setting.

Table: Numerical approximations to the spectral data of $H = \frac{1}{2}(P^2 + Q^2)$ over $\mathbf{Q}_3[\sqrt{3}]$.

Theoretical eigenvalue	Numerical eigenvalue	Theoretical multiplicity	Numerical multiplicity	Type of eigenfunction	Comment
$0 < \lambda_0 < 9/13 \approx 0.6923$	0.6684	1	1	radial	
?	4.6922	?	1	radial	
?	4.7158	?	1	radial	
5	5.0000	2	2	shell function	$2 = 1 + 1$: Coming from two different shells.
9	9.0000	4	4	shell function	All supported on the same shell.
?	40.5213	?	2	radial	
$40 + 5/9 = 40.5555\ldots$	40.5555	2	2	shell function	$2 = 1 + 1$: Coming from two different shells.
41	41.0000	8	8	shell function	$8 = 4 + 4$: Coming from two different shells.
45	45.0000	24	24	shell function	$24 = 12 + 12$: Coming from two different shells.

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The following page is extracted from [BD15]. The function values 0 in columns 2 and 3 are rounded values of numbers of the order 10^{-16} or smaller (machine accuracy).

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TABLE 2. Eigenfunctions for three different eigenvalues, 28 values for each function, coming from all the 5 shells. Both kinds of eigenfunctions occur (shell functions and radial functions). – Shell no. k ($k = 2, 1, 0, -1, -\infty$) is the shell $|x| = 3^k$ (so shell no. $-\infty$ is the shell $|x| = 3^{-\infty} = 0$).

Eigenfunction for the lowest eigenvalue $\boxed{\lambda \approx 0.6684}$. It exhibits a perfect radial behavior. Notice also that the function is strictly positive, in accordance with the corresponding statement for the case $K = \mathbf{Q}_p$ in [VVZ94, p. 186].		Eigenfunction for $\boxed{\lambda = 5}$. Eigenfunctions here are linear combinations of shell functions from two different shells (shells 1 and 0). As should be expected, the function below exhibits non-radial behavior, being non-constant on each shell where it doesn't vanish (shells 1 and 0).		Eigenfunction for $\boxed{\lambda = 9}$. It exhibits a perfect shell function behavior, with support on shell no. 1.	
	Shell no.		Shell no.		Shell no.
$3.5818432 \cdot 10^{-1}$	$-\infty$	0	$-\infty$	0	$-\infty$
$5.5430722 \cdot 10^{-5}$	2	0	2	0	2
$5.5430722 \cdot 10^{-5}$	2	0	2	0	2
$5.5430722 \cdot 10^{-5}$	2	0	2	0	2
$5.5430722 \cdot 10^{-5}$	2	0	2	0	2
$5.5430722 \cdot 10^{-5}$	2	0	2	0	2
$5.5430722 \cdot 10^{-5}$	2	0	2	0	2
$5.5430722 \cdot 10^{-5}$	2	0	2	0	2
$5.5430722 \cdot 10^{-5}$	2	0	2	0	2
$5.5430722 \cdot 10^{-5}$	2	0	2	0	2
$5.5430722 \cdot 10^{-5}$	2	0	2	0	2
$5.5430722 \cdot 10^{-5}$	2	0	2	0	2
$5.5430722 \cdot 10^{-5}$	2	0	2	0	2
$5.5430722 \cdot 10^{-5}$	2	0	2	0	2
$5.5430722 \cdot 10^{-5}$	2	0	2	0	2
$5.5430722 \cdot 10^{-5}$	2	0	2	0	2
$5.5430722 \cdot 10^{-5}$	2	0	2	0	2
$5.5430722 \cdot 10^{-5}$	2	0	2	0	2
$5.5430722 \cdot 10^{-5}$	2	0	2	0	2
$1.2747433 \cdot 10^{-2}$	1	$-2.3459638 \cdot 10^{-1}$	1	$5.9907185 \cdot 10^{-2}$	1
$1.2747433 \cdot 10^{-2}$	1	$2.3459638 \cdot 10^{-1}$	1	$-4.1084268 \cdot 10^{-1}$	1
$1.2747433 \cdot 10^{-2}$	1	$-2.3459638 \cdot 10^{-1}$	1	$-1.0595734 \cdot 10^{-1}$	1
$1.2747433 \cdot 10^{-2}$	1	$2.3459638 \cdot 10^{-1}$	1	$2.7644342 \cdot 10^{-2}$	1
$1.2747433 \cdot 10^{-2}$	1	$-2.3459638 \cdot 10^{-1}$	1	$4.6050157 \cdot 10^{-2}$	1
$1.2747433 \cdot 10^{-2}$	1	$2.3459638 \cdot 10^{-1}$	1	$3.8319834 \cdot 10^{-1}$	1
$3.1960943 \cdot 10^{-1}$	0	$3.9500330 \cdot 10^{-2}$	0	0	0
$3.1960943 \cdot 10^{-1}$	0	$-3.9500330 \cdot 10^{-2}$	0	0	0
$3.5768544 \cdot 10^{-1}$	-1	0	-1	0	-1

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